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Problem 139. **Proposed by Titu Andreescu**, **American Mathematics Competitions**.
(Text of the problem corrected by Giovanni Parzaneze.)

The sequence (a_n) satisfies $a_0 = 1, a_1 = 0, a_2 = 1, a_3 = 16,$ and

$a_{n+2} - a_{n-2} = 18(a_{n+1} - a_{n-1})$ for all $n \geq 2.$

Prove that a_n is a perfect square for all n.

Solution by Arkady Alt , **San Jose** ,**California**, **USA**.

First we will prove that sequence (a_n) can be defined by recurrence of the second order and will do it in two steps.

1. Since $a_{n+2} - a_{n-2} = 18(a_{n+1} - a_{n-1}) \Leftrightarrow a_{n+2} - 17a_{n+1} - 17a_n + a_{n-1} =$
$a_{n+1} - 17a_n - 17a_{n-1} + a_{n-2}$ for any $n \geq 2$ then
$a_{n+2} - 17a_{n+1} - 17a_n + a_{n-1} = a_3 - 17a_2 - 17a_1 + a_0 = 0$ for any $n \in \mathbb{N}$.

2. Noting that $a_{n+2} - 17a_{n+1} - 17a_n + a_{n-1} = 0 \Leftrightarrow$
$a_{n+2} - 18a_{n+1} + a_n + a_{n+1} - 18a_n + a_{n-1} = 0 \Leftrightarrow$
$a_{n+2} - 18a_{n+1} + a_n = -(a_{n+1} - 18a_n + a_{n-1}), \forall n \in \mathbb{N}$ we obtain
$a_{n+1} - 18a_n + a_{n-1} = (-1)^{n-1}(a_2 - 18a_1 + a_0) = 2(-1)^{n-1}, \forall n \in \mathbb{N},$

Thus, sequence (a_n) is defined by recurrence

(**1**) $a_{n+1} - 18a_n + a_{n-1} = 2(-1)^{n-1}, n \in \mathbb{N}$ and initial conditions $a_0 = 1, a_1 = 0.$

To prove that a_n is a perfect square for all n we will find sequence (b_n) such that $a_n = b_n^2$ which is defined by recurrence $b_{n+1} - pb_n + qb_{n-1} = 0, n \in \mathbb{N}$ with initial conditions $b_0 = 1, b_1 = 0.$

Since $b_n b_{n+2} - b_{n+1}^2 = b_n(pb_{n+1} - qb_n) - b_{n+1}(pb_n - qb_{n-1}) = q(b_{n-1}b_{n+1} - b_n^2), n \in \mathbb{N}$
then $b_{n-1}b_{n+1} - b_n^2 = q^{n-1}(b_0 b_2 - b_1^2) = q^{n-1}b_2 = q^{n-1}(pb_1 - qb_0) = -q^{n-1}$

Hence $(b_{n+1} + qb_{n-1})^2 = p^2 b_n^2 \Leftrightarrow (b_{n+1} + qb_{n-1})^2 - p^2 b_n^2 = 0 \Leftrightarrow$
$b_{n+1}^2 - p^2 b_n^2 + q^2 b_{n-1}^2 + 2qb_{n-1}b_{n+1} = 0 \Leftrightarrow b_{n+1}^2 - p^2 b_n^2 + q^2 b_{n-1}^2 + 2q(b_n^2 - q^{n-1}) \Leftrightarrow$
$b_{n+1}^2 - (p^2 - 2q)b_n^2 + q^2 b_{n-1}^2 = 2q^{n-1}$. Claims $p^2 - 2q = 18$ and $q^{n-1} = (-1)^{n-1}$ gives us $q = -1, p = 4$ and we obtain recurrence $b_{n+1} - 4b_n - b_{n-1} = 0, n \in \mathbb{N}$ with initial condition $b_0 = 1, b_1 = 0.$ Since $b_n \in \mathbb{Z}$ and $(b_n^2), (a_n)$ satisfy to the same recurrence and $b_0^2 = a_0 = 1, b_1^2 = b_1 = 0$ then by Math Induction we obtain
$a_n = b_n^2$ for any $n \in \mathbb{N} \cup \{0\}.$